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On Controllability Problems of High-Order Dynamical Multi-Agent Systems

Ning Cai · Junwei Cao · Minghua Liu · Haiying Ma

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Abstract Controllability problems of dynamical high-order multi-agent systems are concerned. A methodology on graph topology transformation is proposed, which can help to analyze the controllability. According to this methodology, a path can be regarded as controllability canonical form of graph, and any controllable graph can be transformed into an equivalent path. Besides, controllability of heterogeneous multi-agent systems is also discussed.

Keywords Controllability · Multi-agent system · Graph topology

الخلاصة

تم الاهتمام بمشاكل قابلية التحكم لأنظمة متعددة الوكيل ذات الترتيب العالي. وعرضت منهجية لنقل تخطيط الرسم البياني التي يمكن أن تساعد في تحليل قابلية التحكم. ويمكن - وفقاً لهذه المنهجية - اعتبار مسار ما على أنه شكل معياري لقابلية التحكم بالرسم البياني، كما يمكن نقل أي رسم بياني قابل للتحكم إلى مسار مكافئ. وتم - إضافة إلى ذلك - مناقشة قابلية التحكم بنظم متعددة الوكيل وغير متجانسة.

1 Introduction

During recent years, dynamical multi-agent systems have been extensively studied by scholars in the field of control theory [1–22]. The majority of attention is focused on the consensus problem [3–9], which is essentially a stability problem [7]. However, only a few researchers have started to notice the controllability problems. Mesbahi [10] proposed the concept "state-dependent graph" and defined "controllability" for state-dependent graphs. Tanner [11] postulated that more information exchange may be detrimental to controllability. Ji et al. [12] gave some sufficient conditions for controllability, based on the algebraic characteristics of certain matrices about the graphs. Rahmani et al. [13] extended the work of Ji et al. in [12], concerning the relationship between graph symmetry and controllability. Cai et al. [14–18] studied formation controllability of high-order systems and discussed approaches for controllability improvement of graph topology. Liu et al. [19,20] concerned the controllability of discrete-time multi-agent systems with switching graph topologies, applying their previous results on controllability of switching linear systems. Wang et al. [21] attempted to discuss the effect on controllability through different schemes of leader choices and strength of communication links. Ji et al. [22] proposed conditions for graph controllability, which are analogous to the early results in [23]. Liu et al. [24] endeavored to analyze the relation between controllability and topology of large-scale weighted complex networks, from a viewpoint upon multiple disciplinary backgrounds.

The results in the current paper form an extension to our previous works [15–18]. The novel features here are twofold: (1) A methodology is introduced for controllability analysis, which can transform a graph topology into equivalent forms. Based on this methodology, it will be shown in detail that any controllable graph with a single leader can be transformed into a graph. (2) The controllability problem of heterogeneous multi-agent systems is also discussed.

The controllability analysis of multi-agent systems is far more involved than stability analysis. The major concern of

N. Cai (✉) · M. Liu
School of Electrical Engineering, Northwest University for Nationalities, Lanzhou 730030, China
e-mail: caining91@tsinghua.org.cn

J. Cao
Research Institute of Information Technology, Tsinghua University, Beijing 100084, China

H. Ma
School of Economics, Northwest University for Nationalities, Lanzhou 730030, China

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nearly all recent relevant works [11–13,19–24] is to attempt to reveal the relationship between the geometric characteristics of a graph topology and its controllability. They did obtain some theoretical results. However, none of them have really finished their initial objective. Actually, such a direct relationship might not exist at all. Our technical route for investigating the controllability of multi-agent systems is different from the majority of relevant works, while we mainly focus on the algebraic characteristics of matrices instead of paying attention to the topologies of graphs.

The rest of this paper is organized as follows. Section 2 formulates the controllability problem of high-order multi-agent systems. Section 3 introduces the graph transformation method, providing one of the main results. Based on it, the controllability canonical form of graph is also analyzed. Section 4 discusses the controllability problem for heterogeneous systems. Finally, Sect. 5 concludes the paper.

Notation: ϕ represents any vector with all 1 elements. P_N denotes an Nth order path.

2 Problem Formulation

The dynamical multi-agent system model to be considered is described as:

$$\dot{x}_i = F\sum_{j=1}^{N} w_{ij}(x_j - x_i) + Bu_i \ (i = 1, 2, \ldots, N) \tag{1}$$

where $x_i \in R^d$, $F \in R^{d\times d}$, $w_{ij} \in R$, $B \in R^{d\times m}$, and $u_i \in R^m$. x_i is the state vector of an agent. w_{ij} is the edge or arc weight of a graph G, representing the strength of information link between the two neighbors i and j. For an LTI system, the time-invariant graph G can be denoted by its adjacency matrix W. If the input $u_i(t) \equiv 0$, then agent i is called a *follower*; or if agent i is actuated by some $u_i(t)$ not being zero identically, it is called a *leader*. If all state vectors of agents are stacked together, then the entire state matrix of the system is $X \in R^{d\times N}$.

The first-order dynamical multi-agent system model with linear consensus protocol that has been studied extensively in the literature, e.g., in [3,4,19–24], is a particular instance of the model depicted by (1).

The dynamics of the overall system can be described by the following matrix state equation:

$$\dot{X} = -FXL^{\mathrm{T}} + BU \tag{2}$$

where $L \in R^{N\times N}$ is the Laplacian matrix of the graph G. The relationship between the adjacency and the Laplacian matrices of a graph G is:

$$L = \mathrm{diag}(W\phi) - W. \tag{3}$$

From the definition of L, one will find that:

$$L\phi = 0, \tag{4}$$

i.e., ϕ is always an eigenvector corresponding to eigenvalue 0 of L. (4) can be regarded as a criterion to check whether a given matrix is a Laplacian matrix of certain existing graph.

Remark 1 The matrix differential equation (2) is more complicated than the usual vector equation studied in control theory. There is not only a left coefficient matrix $-F$, but also an additional right coefficient matrix L^{T}. Thus, it is more difficult to analyze the controllability of these systems.

Suppose that N_{l} of the agents are leaders. To discriminate between the followers and the leaders, partition L:

$$L = \begin{bmatrix} L_{\mathrm{ff}} & L_{\mathrm{fl}} \\ L_{\mathrm{lf}} & L_{\mathrm{ll}} \end{bmatrix} \tag{5}$$

where $L_{\mathrm{ff}} \in R^{(N-N_{\mathrm{l}})\times(N-N_{\mathrm{l}})}$ indicates the arcs of G among the followers; $L_{\mathrm{fl}} \in R^{(N-N_{\mathrm{l}})\times N_{\mathrm{l}}}$ the arcs from the leaders to the followers; $L_{\mathrm{lf}} \in R^{N_{\mathrm{l}}\times(N-N_{\mathrm{l}})}$ the arcs from the followers to the leaders; and $L_{\mathrm{ll}} \in R^{N_{\mathrm{l}}\times N_{\mathrm{l}}}$.

Definition 1 (*Graph Controllability* [14,22]) With the last N_{l} agents as the leaders and a partitioned form of adjacency matrix L as (5), the graph G of the dynamical multi-agent system is *controllable* if and only if $(L_{\mathrm{ff}}, L_{\mathrm{fl}})$ is controllable.

With the criterion given by the following lemma, complete controllability of an LTI dynamical multi-agent system of high order can be checked.

Lemma 1 [14,18] *The LTI dynamical multi-agent system (1) is completely controllable if and only if the two conditions below are simultaneously satisfied:*

(1) The graph G is controllable;
(2) (F, B) is controllable.

Evidently, Lemma 1 manifests a separation principle, with the two conditions being independent of each other. The controllability of a multi-agent system (1) is determined jointly by the controllability of the two dynamic systems below:

$$\dot{\xi} = -L\xi \tag{6}$$

$$\dot{\eta} = F\eta + Bu \tag{7}$$

The following assumption is assumed to be satisfied throughout this paper. Under this assumption, the rest of the problem for controllability of (1) is determined by the controllability of graph, i.e., controllability of system (6).

Assumption 1 (F, B) is controllable.

Remark 2 The settings of model and problems introduced in this section are mainly similar to that in [14]. Based on the train of thoughts, new approaches and results will be expounded in the subsequent two sections.

3 Equivalent Transformation to Graphs

The mapping between a weighted digraph G and a square matrix $W(G)$ is bijective: $W(G) \leftrightarrow G$. Nevertheless, Laplacian matrix cannot reflect the relationship between any vertex and itself. Actually, if we assume that there is no loop in G, a graph topology can also be uniquely determined by its Laplacian matrix according to the following equation:

$$W(G) = D(G) - L(G) \tag{8}$$

where $D(G)$ is the diagonal of $L(G)$.

It is evident that Laplacian matrix plays a major role in regulating the dynamics of dynamical multi-agent systems. Consider the dynamic system (6). If a similarity transformation is applied to matrix L, e.g., $\tilde{L} = P^{-1}LP$, an equivalent dynamic equation $\dot{\tilde{\xi}} = PLP^{-1}\tilde{\xi}$ can be derived. The similar matrix $\tilde{L}$ of L is the Laplacian matrix representing another graph topology $\widehat{G}$, which can be regarded as being similar to the original graph G in the sense of system dynamics.

Although such a $\widehat{G}$ is similar to G, the graph topology is different. If G is bidirectional, its Laplacian matrix is symmetric. But generally the symmetry of a matrix could not be preserved after a similarity transformation. Therefore, $\widehat{G}$ might become a directed graph. There is a determinate and clear definition as to the Laplacian matrix of a bidirectional graph. But for directed graphs, there are different definitions. Instances over three types of definitions exist in the literature [3,25–27]. In this paper, the simplest definition in [3] is taken by default. Despite the variation of definitions, usually (4) is a fundamental rule for Laplacian matrix that zero must be the eigenvalue for eigenvector. Unfortunately, this rule might be broken after a direct linear transformation, and $\tilde{L}\phi = P^{-1}LP\phi$ is no longer zero. Only specific non-singular transformations could reserve this property.

Proposition 1 *If the non-singular transformation P is stochastic [28], then for the similar matrix $\tilde{L}$, zero is still the eigenvalue of eigenvector ϕ. Thus, such an $\tilde{L}$ is the Laplacian matrix of a graph topology.*

Proof Because P is stochastic, $P\phi = \phi$. Therefore,

$$\tilde{L}\phi = P^{-1}LP\phi = P^{-1}L\phi = P^{-1}0 = 0.$$

□

As an instance, the most typical stochastic matrix is permutation matrix [28]. Actually, a similarity transformation via a permutation matrix P transforms an original graph G into its isomorphic graph $\widehat{G}$.

In general, the result of a similarity transformation to some Laplacian matrix L does not meet condition (4) anymore. $\tilde{L}$ can be regarded as the sum of two parts: $\tilde{L} = \widehat{L} + \Delta\widehat{L}$, with $\widehat{L}$ representing the Laplacian matrix derived from the non-diagonal entries of $\tilde{L}$ satisfying condition (4), and $\Delta\widehat{L}$ is the redundant part, which is diagonal. If $\Delta\widehat{L}$ is not zero, it can be regarded that there is an adjunctive virtual vertex in the graph increasing its order into $N + 1$, with the state of the corresponding virtual agent always being 0. Meanwhile, $\Delta\widehat{L}$ indicates the neighborship between each agent and this particular virtual agent.

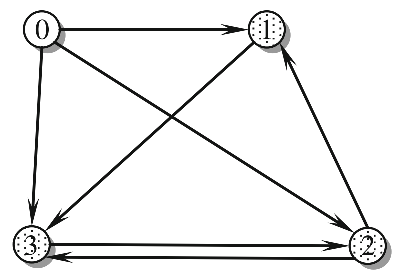

Fig. 1 Third-order graph induced by generalized Laplacian matrix

Example 1 Let

$$\tilde{L} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix}.$$

According to the definition of Laplacian matrix (3),

$$\widehat{L} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ -1 & -1 & 2 \end{bmatrix}$$

and

$$\Delta\widehat{L} = \tilde{L} - \widehat{L} = \mathrm{diag}(\begin{bmatrix} 1 & -1 & -4 \end{bmatrix})$$

The adjacency matrix $W(\widehat{G})$ is

$$W(\widehat{G}) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix}.$$

The transformed $\widehat{G}$ is shown in Fig. 1.

In a dynamical multi-agent system described by (2), if some of the agents are assigned as leaders, with the Laplacian matrix in the partitioned form $L(G) = \begin{bmatrix} L_{ll} & L_{lf} \\ L_{fl} & L_{ff} \end{bmatrix}$, then the leader set and follower set are determinate, and any linear transformation to the graph topology would not affect this situation. Therefore, a feasible non-singular linear transformation should take the form: $P = \begin{bmatrix} P_l & 0 \\ 0 & P_f \end{bmatrix}$, with both $P_l \in R^{N_l \times N_l}$ and $P_f \in R^{(N-N_l)\times(N-N_l)}$ being non-singular. So we have:

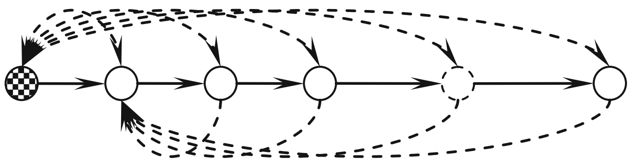

Fig. 2 Single-leader controllability canonical digraph, (Left most agent is the leader while others are followers; *dashed arcs* may exist or not)

$$\tilde{L} = P^{-1}LP = \begin{bmatrix} P_l^{-1} & 0 \\ 0 & P_f^{-1} \end{bmatrix} \begin{bmatrix} L_{ll} & L_{lf} \\ L_{fl} & L_{ff} \end{bmatrix} \begin{bmatrix} P_l & 0 \\ 0 & P_f \end{bmatrix} = \begin{bmatrix} P_l^{-1} L_{ll} P_l & P_l^{-1} L_{lf} P_f \\ P_f^{-1} L_{fl} P_l & P_f^{-1} L_{ff} P_f \end{bmatrix}$$

After the technique for linear transformation to graphs is introduced, one will see its application in controllability analysis. By the theorem that follows, it will be shown that a path is representative as the controllable type of graph topology.

Lemma 2 [14] *A path P is strictly structurally controllable.*

So long as one end of the path is the single leader, and the path is connected, whatever the values of edge weights, the entire path must be controllable.

Definition 2 (*Controllability Canonical Digraph with Single Leader*) Suppose there is a dynamical multi-agent system with N agents indexed by $\{1, 2, \ldots, N\}$, with agent N the single leader, and G a weighted digraph. G is a *controllability canonical digraph* if its Laplacian matrix $L(G) \in R^{N\times N}$ has the following form:

$$\begin{bmatrix} 1 & -1 & & \cdots & 0 & 0 \\ & 1 & -1 & \cdots & 0 & 0 \\ & & \cdots & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ * & * & \cdots & \cdots & * & \times \\ * & * & \cdots & \cdots & * & * \end{bmatrix} \tag{9}$$

where '×' is a non-zero value, and '*' could be any value.

In a controllability canonical digraph with single leader, all the followers form a path. The configuration of such a digraph is illustrated in Fig. 2 [14,18].

Lemma 3 [14]*: A single-leader controllability canonical digraph must be a controllable digraph.*

Theorem 1 *If the graph topology of a single-leader dynamical multi-agent system is denoted by digraph G, and G is controllable, then there must exist a non-singular linear transformation* $P = \begin{bmatrix} P_f & 0 \\ 0 & P_l \end{bmatrix} \in R^{N\times N}$, *such that the Laplacian matrix be transformed into* $L(\widehat{G}) = PL(G)P^{-1}$, *with* $L(\widehat{G})$ *representing a controllability canonical digraph, i.e., a path.*

Proof Let $P_f = \begin{bmatrix} p_1 & p_2 & \cdots & p_{N-1} \end{bmatrix}^T \in R^{(N-1)\times(N-1)}$. Because

$$L(\widehat{G}) = PL(G)P^{-1} = \begin{bmatrix} P_f & 0 \\ 0 & P_l \end{bmatrix} \begin{bmatrix} L_{ff} & L_{fl} \\ L_{lf} & L_{ll} \end{bmatrix} \begin{bmatrix} P_f^{-1} & 0 \\ 0 & P_l^{-1} \end{bmatrix} = \begin{bmatrix} P_f L_{ff} P_f^{-1} & P_f L_{fl} P_l^{-1} \\ P_l L_{lf} P_f^{-1} & P_l L_{ll} P_l^{-1} \end{bmatrix} = \begin{bmatrix} \widehat{L}_{ff} & \widehat{L}_{fl} \\ \widehat{L}_{lf} & \widehat{L}_{ll} \end{bmatrix}$$

we have $\widehat{L}_{ff} = P_f L_{ff} P_f^{-1}$ or $P_f L_{ff} = \widehat{L}_{ff} P_f$. According to (9), suppose that the structure of $L(\widehat{G})$ takes the canonical form as

$$\widehat{L}_{ff} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \cdots & \cdots \\ * & * & * & * \end{bmatrix}$$

Then,

$$\begin{bmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_{N-1}^T \end{bmatrix} L_{ff} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \cdots & \cdots \\ * & * & * & * \end{bmatrix} \begin{bmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_{N-1}^T \end{bmatrix} \tag{10}$$

The series of equations below can be derived from the rows of (10) above

$$p_1^T L_{ff} = p_1^T - p_2^T$$
$$p_2^T L_{ff} = p_2^T - p_3^T$$
$$\cdots\cdots$$
$$p_{N-2}^T L_{ff} = p_{N-2}^T - p_{N-1}^T.$$

It follows that

$$\begin{cases} p_2^T = p_1^T (I - L_{ff}) \\ p_3^T = p_2^T (I - L_{ff}) = p_1^T (I - L_{ff})^2 \\ \vdots \\ p_{N-1}^T = p_1^T (I - L_{ff})^{N-2} \end{cases} \tag{11}$$

Let

$$\widehat{L}_{fl} P_l = P_f L_{fl} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} p_1^T L_{fl} \\ p_1^T (I - L_{ff}) L_{fl} \\ \vdots \\ p_1^T (I - L_{ff})^{N-2} L_{fl} \end{bmatrix} \in R^{N-1}$$

Transpose this equation:

$$\begin{aligned}
(\widehat{L}_{\mathrm{fl}} P_{\mathrm{l}})^{\mathrm{T}} &= \begin{bmatrix} p_1^{\mathrm{T}} L_{\mathrm{fl}} \\ p_1^{\mathrm{T}} (I - L_{\mathrm{ff}}) L_{\mathrm{fl}} \\ \dots \\ p_1^{\mathrm{T}} (I - L_{\mathrm{ff}})^{N-2} L_{\mathrm{fl}} \end{bmatrix}^{\mathrm{T}} \in R^{1\times(N-1)} \\
&= \left[\, p_1^{\mathrm{T}} L_{\mathrm{fl}} \;\; p_1^{\mathrm{T}} (I - L_{\mathrm{ff}}) L_{\mathrm{fl}} \;\dots\; p_1^{\mathrm{T}} (I - L_{\mathrm{ff}})^{N-2} L_{\mathrm{fl}} \,\right] \\
&= p_1^{\mathrm{T}} \left[\, L_{\mathrm{fl}} \;\; (I - L_{\mathrm{ff}}) L_{\mathrm{fl}} \;\dots\; (I - L_{\mathrm{ff}})^{N-2} L_{\mathrm{fl}} \,\right] \\
&= \left[\, 0 \; 0 \dots 1 \,\right]. \qquad (12)
\end{aligned}$$

As a result

$$p_1^{\mathrm{T}} = \left[\, 0 \; 0 \dots 1 \,\right] \left[\, L_{\mathrm{fl}} \;\; (I - L_{\mathrm{ff}}) L_{\mathrm{fl}} \;\dots\; (I - L_{\mathrm{ff}})^{N-2} L_{\mathrm{fl}} \,\right]^{-1}.$$

It is evident that if G is controllable, then $[L_{\mathrm{fl}} (I - L_{\mathrm{ff}}) L_{\mathrm{fl}} \dots (I - L_{\mathrm{ff}})^{N-2} L_{\mathrm{fl}}]$ must be non-singular. Besides, all the eigenvectors of $(I - L_{\mathrm{ff}})$ superpose those of L_{ff}. Therefore, the root of equation (12) p_1^{T} must exist. According to (11), it is easy to derive the required similar transformation P_{f}. □

Remark 3 Theorem 1 constitutes one of the main contributions of the current paper. Actually, it is analogous to the corresponding result in linear systems theory that the controllability canonical form of a single-input system is of the upper-triangular formation of series of integrators.

4 Controllability of Heterogeneous LTI Dynamical Multi-Agent Systems

So far, we have dealt with homogeneous multi-agent systems. Suppose the model is heterogeneous, then the dynamics of agents are depicted by the following equations:

$$\dot{x}_i = \sum_{j=1}^{N} w_{ij} F_{ij} (x_j - x_i) + B_i u_i \;\; (i \in \{1, 2, \dots, N\}) \qquad (13)$$

where $x_i \in R^d$, $w_{ij} \in R$, $F_{ij} \in R^{d\times d}$, $u_i \in R^m$ and $B_i \in R^{d\times m}$. The scalar w_{ij} can be combined into F_{ij}, and (13) becomes

$$\dot{x}_i = \sum_{j=1}^{N} F_{ij} (x_j - x_i) + B_i u_i \;\; (i \in \{1, 2, \dots, N\}). \qquad (14)$$

Comparing (14) with the well-known first-order multi-agent system model in the literature [3,4,19–24], one will sense that the square matrix $F_{ij} \in R^{d\times d}$ plays the role of edge weight. If F_{ij} is regarded as edge weight of the graph topology G, then the adjacency matrix is:

$$W = \begin{bmatrix} 0 & F_{12} & \cdots & F_{1N} \\ F_{21} & 0 & \cdots & F_{2N} \\ \dots & \dots & \dots & \dots \\ F_{N1} & F_{N2} & \cdots & 0 \end{bmatrix} \in R^{dN\times dN}. \qquad (15)$$

Each element in such an adjacency matrix is a matrix instead of a scalar value. Naturally, the corresponding Laplacian matrix is:

$$L = \begin{bmatrix} \sum_{j=1}^{N} F_{1j} & -F_{12} & \dots & -F_{1N} \\ -F_{21} & \sum_{j=1}^{N} F_{2j} & \dots & -F_{2N} \\ \dots & \dots & \dots & \dots \\ -F_{N1} & -F_{N2} & \dots & \sum_{j=1}^{N} F_{Nj} \end{bmatrix}. \qquad (16)$$

The analysis for the controllability of such a heterogeneous dynamical multi-agent system is far more sophisticated than homogeneous systems depicted by (1). For heterogeneous systems, two sufficient conditions for uncontrollability will be proposed in this section.

Remark 4 It is well known that [15] for any LTI system in a given coefficient space, there always exist controllable systems in its arbitrarily small neighborhood. Also, for any LTI system with random coefficients, the probability for controllability equals to 1. Consequently, controllable systems are common, whereas uncontrollable systems are exceptional cases. In such a sense, criteria to check uncontrollability are more meaningful than controllability.

Definition 3 (*Identical Uncontrollability*) Suppose that there is a set of matrix pairs $\{[\, A_i \in R^{d\times d} \;\; B_i \in R^{d\times m} \,] \,|\, i = 1, 2, \dots, N\}$, and $B_i = \left[\, b_1^{(i)} \; b_1^{(i)} \cdots b_m^{(i)} \,\right]$, with $b_k^{(i)} \in R^d$ being a column vector. The set of matrix pairs is *identically uncontrollable* if each shares a common left eigenvector $q \in R^d$, and the set

$$\{b_k^{(i)} \,\big|\, i = 1, 2, \dots, N \;\; k = 1, 2, \dots, m\} \subseteq q^{\perp}$$

with $q^{\perp}$ the orthogonal complement of vector q.

Proposition 2 *If the matrix-weighted graph represented by (15) is bidirectional: $F_{ik} = F_{ki}$ ($\forall i, k = 1, 2, \dots, N$), and the set of matrix pairs $\{[\, F_{ik} \;\; B_i \,] \,|\, i, k = 1, 2, \dots, N\}$ is identically uncontrollable, then the multi-agent system (14) is uncontrollable.*

Proof The system state is first written in vector form as:

$$x = \mathrm{vec}(X) = \left[\, x_1^{\mathrm{T}} \; x_2^{\mathrm{T}} \cdots x_N^{\mathrm{T}} \,\right]^{\mathrm{T}} \in R^{dN}$$
$$[u = \mathrm{vec}(U) = \left[\, u_1^{\mathrm{T}} \; u_2^{\mathrm{T}} \cdots u_N^{\mathrm{T}} \,\right]^{\mathrm{T}} \in R^{mN}$$

Then, the dynamics of the entire system can be expressed by equation:

$$\dot{x} = -Lx + Bu$$

where $L \in R^{dN \times dN}$ takes the form of (16), and

$$B = \begin{bmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_N \end{bmatrix} \in R^{dN \times mN}.$$

Suppose $q \in R^d$ is the specific common left eigenvector of F_{ik} : $q^{\mathrm{T}} F_{ik} = \lambda_{ik} q^{\mathrm{T}}$, with $\lambda_{ik} \in C$ the corresponding eigenvalue. By considering the structure of L according to (16) and the identical uncontrollability, it is simple to verify that vector $(\phi \otimes q) \in R^{dN}$ must simultaneously satisfy both of the equations below:

$$\mathrm{G} \begin{cases} (\phi \otimes q)^{\mathrm{T}} L = 0 \\ (\phi \otimes q)^{\mathrm{T}} B = 0 \end{cases}$$

where $\phi = \begin{bmatrix} 1 \ 1 \ \cdots \ 1 \end{bmatrix}^{\mathrm{T}} \in R^N$. According to the PBH test for LTI systems, the proof can be concluded. □

Proposition 3 *If the graph topology G of the heterogeneous multi-agent system is structurally uncontrollable [29], then the overall system must be uncontrollable.*

Proof Since the graph topology is structurally uncontrollable, the configuration of information flow among agents must be non-input-connectable [29]. According to the definition of input connectability, some of the follower agents receive no information transmitted from any leader, either directly or indirectly, and they are out of control. As a result, the entire multi-agent system is uncontrollable. □

5 Conclusions

Controllability problem of high-order LTI multi-agent systems is the major subject in this paper. An approach based on linear transformation to Laplacian matrix is devised, which can transform the graph topology of a dynamical multi-agent system into another graph equivalent to the original one in the sense of swarm dynamics, but is more convenient for analysis. Based on this approach and our previous theoretical results, it is shown that a path is a controllability canonical form for graphs and is strictly structurally controllable. Controllability problem of heterogeneous LTI high-order multi-agent systems was also dealt with, which is far more sophisticated than homogeneous situations. For heterogeneous systems, two sufficient conditions for uncontrollability were presented, concerning the graph topology and the interactive dynamics, respectively. The controllability canonical form of digraph in this paper has only one leader. Besides, the current conditions for uncontrollability of heterogeneous dynamical multi-agent systems are relatively conservative. The study on heterogeneous systems is still at its initial stage. Relaxing these limitations will be the main objective of our future research.

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